

UNCLASSIFIED

AD 4 2 2 8 4 3

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

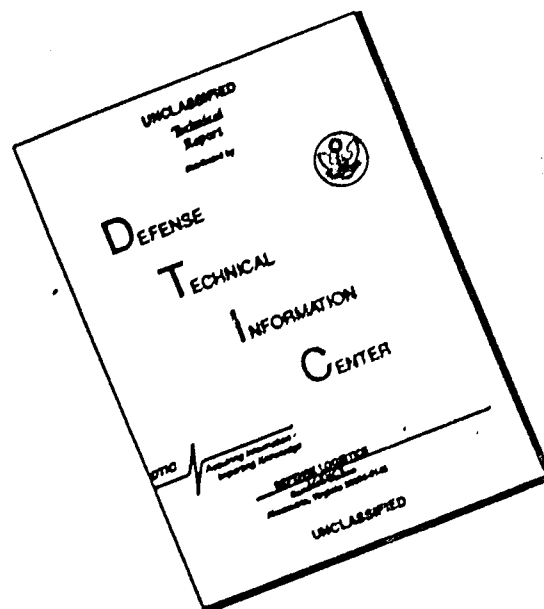
CAMERON STATION, ALEXANDRIA, VIRGINIA



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

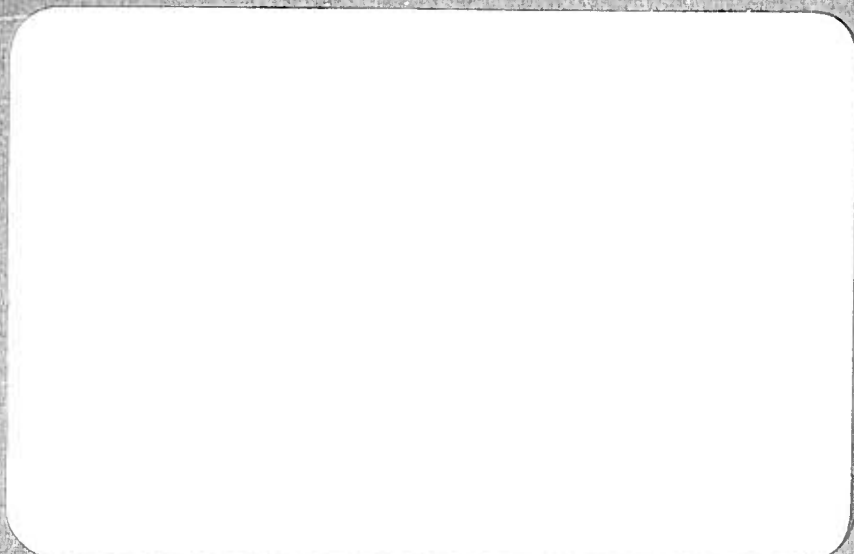
AD No. _____

DDC FILE COPY

422843

(5) 739 200

①



DDC
NOV 15 1963

1151A A

(5)

139000

(6)

THE SIMPLEX METHOD,

(10) by

G. B. Dantzig,

P-891

July 9, 1956

(B)

The RAND Corporation

1700 MAIN ST. • SANTA MONICA • CALIFORNIA

SUMMARY

△ The simplex method of solution of a linear program first transforms the original system to an equivalent system of m equations in canonical form by an elimination of m of the n unknowns. If the right choice of m variables is made, then by equating the remaining variables to zero, an optimal solution is obtained to the original problem. If not, the method produces an "improved" set of m variables and a corresponding canonical form. The procedure is iterated until an optimum solution is obtained.

4

CONTENTS

SUMMARY	11
Section	
I. ALGEBRA OF THE SIMPLEX METHOD	1
Introduction	1
The Problem	2
The Canonical Form	3
The Initial Basic Feasible Solution and Test for Optimality	5
Improving a Non-Optimal Basic Feasible Solution	6
Iterative Procedure	10
Degeneracy	10
Appendix: Systems of Linear Equations	12
Equivalent Systems	12
Elementary Operations	14
Equivalent Triangular and Canonical m x m Systems	14
Reduction of m by n system to an equivalent canonical system	16
II. THE SIMPLEX TABLEAU: AN EXAMPLE	18
III. FINDING AN INITIAL BASIC FEASIBLE SOLUTION ...	22

THE SIMPLEX METHOD

I. ALGEBRA OF THE SIMPLEX METHOD

Introduction

"Simplex" is a mathematical term meaning a k -dimensional "triangle"; i.e., it is the generalization of the word triangle for two dimensions or of a tetrahedron for three dimensions. The term "simplex method" is derived from one of several geometric interpretations of the technique in which this figure plays a role. In this lecture its mathematical characteristics will be developed. It will be shown that, except possibly for a member of a class of cases called "degenerate," the simplex algorithm will produce in a finite number of iterations a solution (if one exists) which minimizes a linear form in non-negative variables subject to a system of linear equations. The excepted class, which is important from the viewpoint of theory (rather than practice) will not be considered here. However the rule of random choice given for resolving the degenerate case can be shown to lead to an optimal solution in a finite number of iterations with probability one.

The chief feature of the method is that it calls for elimination of variables from the equations in a manner quite analogous to ordinary (Gaussian) elimination for solving a simultaneous system involving m equations in m unknowns. After an elimination, using some selected set of m variables, a solution is obtained by setting all remaining variables equal to zero. If this

solution is feasible it can be used as a starting solution for the standard simplex algorithm. The next step is a test to determine whether it is a minimum solution. If not, a new variable is chosen to be eliminated and one of those variables previously selected is dropped from the selected set. This will give rise to a new feasible solution whose cost z is lower than the previous solution (in the case of degeneracy referred to earlier, it may have the same cost). The procedure may be iterated and it can be shown that in a finite number of steps an optimal solution can be obtained (if one exists). Since it is necessary to have an initial solution that is feasible, the procedure is divided into two phases: In phase I a starting feasible solution (if it exists) is obtained for phase II.

The Problem

Find values of x_1, x_2, \dots, x_n satisfying the simultaneous system of equations

$$\begin{aligned} & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ (1) \quad & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \end{aligned}$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

and the inequalities

$$(2) \quad x_j \geq 0 \quad (j = 1, 2, \dots, n)$$

which minimize the linear form

The Canonical Form

Let us suppose that it has been determined that a starting feasible solution can be obtained using some known set of m of the n variables. By relabeling the subscripts the selected variables can be arranged to be (x_1, x_2, \dots, x_m) . The first step is to eliminate x_1 from the cost form and from all but one of the equations (1). By relabeling the subscripts designating the different equations the equation in which x_1 has not been eliminated can be arranged to be the first. The second step is to try to eliminate x_2 from the modified cost form, the first equation, and all but one of the modified set of equations $i = 2, \dots, m$. If possible, this gives rise to a second set of modified equations and a cost form in which x_1 appears only in the first equation and, by relabeling, x_2 appears only in the second equation. By continuing the elimination in this manner, if possible, for variables x_3, \dots, x_m in turn, the final modified system has the "canonical" form

$$(4) \quad \begin{array}{l} x_1 + \bar{a}_{1m+1}x_{m+1} + \dots + \bar{a}_{1j}x_j + \dots + \bar{a}_{1n}x_n = \bar{b}_1 \\ x_2 + \bar{a}_{2m+1}x_{m+1} + \dots + \bar{a}_{2j}x_j + \dots + \bar{a}_{2n}x_n = \bar{b}_2 \\ \vdots \\ x_m + \bar{a}_{m,m+1}x_{m+1} + \dots + \bar{a}_{mj}x_j + \dots + \bar{a}_{mn}x_n = \bar{b}_m \\ \bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_jx_j + \dots + \bar{c}_nx_n = z - z_0 \end{array}$$

where \bar{a}_{1j} , \bar{c}_j , \bar{b}_1 are the modified values of a_{1j} , c_j , b_1 due to the eliminations and z_0 is some constant. (Problem: Prove the values of \bar{a}_{1j} are independent of the order of elimination.)

In certain cases it is not possible to perform the elimination in the manner prescribed and achieve the canonical form. This would be the case if there are:

- (a) redundant equations;
- (b) inconsistent equations;

and may be the case for

- (c) an arbitrary choice of the set of m variables.

Since methods we shall discuss for solving the L.P. problem depend on standard methods for solving a system of linear equations, an appendix has been added to the end of this Section for review of properties of such systems.

A set of m variables is called a "basic set" if it is possible to use them to reduce the system of equations to the canonical form. There are three ways usually employed for determining whether a given set of variables is a basic set:

- (a) Showing the determinant of the coefficients of the selected variables is non-vanishing;
- (b) Showing a set of values of the selected variables can be determined (necessarily unique) whatever be the values of the right-hand side;
- (c) Trying to carry out the eliminations in the manner prescribed.

Problem: Show that if either (a) or (b) or (c) is true the

other two are true also. Prove necessity of uniqueness in (b).

The Initial Basic Feasible Solution and Test for Optimality

The standard simplex method works only with basic solutions which are feasible. By a "basic solution" is meant a solution of the constraint equations (1) obtained by equating all non-basic variables to zero.* In the canonical form the basic solution is obvious—namely

$$(5) \quad x_1 = \bar{b}_1, \quad x_2 = \bar{b}_2, \quad \dots, \quad x_m = \bar{b}_m$$

with a cost

$$(6) \quad z = z_0.$$

It is a basic feasible solution if the values of the basic variables are non-negative:

$$(7) \quad \bar{b}_1 \geq 0, \quad \bar{b}_2 \geq 0, \quad \dots, \quad \bar{b}_m \geq 0.$$

In canonical form it is also easy to determine whether a basic feasible solution is optimal. First we introduce some terminology. We shall use the term "cost factor" to denote the coefficients c_j in the cost or objective form (3). We shall use the term "relative cost factor" to denote the coefficient \bar{c}_j of the variable x_j in the cost or objective form of the canonical system. It is relative because it depends on the choice of the basic set

*A variant of this procedure can be developed in which non-basic variables are equated to constant values other than zero.

of variables. In particular, the relative cost factors of the basic variables are always zero.

Theorem 1:

A basic feasible solution is a minimal feasible solution with total cost z_0 if all relative cost factors are non-negative:

$$(8) \quad \bar{c}_j \geq 0 \quad (j = 1, 2, \dots, n).$$

Theorem 2:

Given a minimal basic feasible solution with relative cost factors $\bar{c}_j \geq 0$, then any other feasible solution (not necessarily basic) with the property that $x_j = 0$ for all $\bar{c}_j > 0$ is also a minimal solution; moreover one with the property that at least one $x_j > 0$ for some $\bar{c}_j > 0$ can not be a minimal solution.

Corollary:

A basic feasible solution is the unique minimal feasible solution if all $\bar{c}_j > 0$ for non-basic variables.

Referring to the canonical form it is obvious that if the coefficients of the cost form are all positive or zero, the smallest value of the left-hand side is zero for non-negative x_j . Thus, the smallest value of $z - z_0$ is zero and hence

$$\text{Min } z = z_0$$

and the solution is optimal. Problem: Prove the other theorems.

Improving a Non-Optimal Basic Feasible Solution

On the other hand, if at least one relative cost is negative,

it is possible to construct a new basic feasible solution with a total cost lower than $z = z_0$. (There is exception to this statement for certain "degenerate" cases that we shall discuss later.) Indeed we can obtain a lower cost solution by increasing the value of one of the non-basic variables x_s and adjusting the values of the basic variables accordingly where x_s is any variable whose relative cost factor \bar{c}_s is negative. In particular the index s can be chosen such that

$$(9) \quad \bar{c}_s = \text{Min } \bar{c}_j < 0.$$

The latter is the rule for choice of s followed in practical computational work because it has been found that it usually leads to fewer iterations of the algorithm.

Using the canonical form, let us construct a solution in which x_s takes on some positive value, the values of all other non-basic variables are still zero, and the values of the basic variables are adjusted to take care of the increase in x_s :

$$(10) \quad \begin{aligned} x_1 &= \bar{b}_1 - \bar{a}_{1s}x_s \\ x_2 &= \bar{b}_2 - \bar{a}_{2s}x_s \\ &\dots \dots \dots \\ x_m &= \bar{b}_m - \bar{a}_{ms}x_s. \end{aligned}$$

The total cost associated with this solution is

$$(10.1) \quad z = z_0 + \bar{c}_s x_s.$$

It is clear, since \bar{c}_s has been chosen negative, that the larger the value of x_s the smaller will be the value of z . The only

thing that prevents setting x_s infinitely large is the possibility that the value of one of the basic variables in (10) will become negative. Indeed the following theorem is clear.

Theorem 3:

If for some s , all coefficients of x_s in the canonical system are negative (or zero), including its relative cost factor, then a class of feasible solutions can be constructed such that $z \rightarrow -\infty$ as $x_s \rightarrow +\infty$.

On the other hand, if at least one \bar{a}_{1s} is positive, it will not be possible to increase the value of x_s indefinitely, because beyond the ratio \bar{b}_1/\bar{a}_{1s} , the value of x_1 will be negative. If \bar{a}_{1s} is positive for several i , then the smallest of such ratios, whose subscript will be denoted by $i = r$, will determine the largest value of x_s possible such that all values of x_i in (10) remain non-negative. Let $x_s^* = \max x_s$ possible; then

$$(11) \quad x_s^* = \frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{\substack{\bar{a}_{is} > 0}} \frac{\bar{b}_i}{\bar{a}_{is}} \geq 0$$

where r , in case of a tie, may be chosen as the one with the smallest index or at random from among those tied.

Theorem 4:

If in a basic feasible solution the values of all basic variables are strictly positive and if for some s , the relative cost factor is negative and at least one other coefficient \bar{a}_{1s} of x_s in the canonical system is positive, then a new basic solution can be constructed with lower total cost z .

Indeed it follows from the assumption that all $\bar{b}_i > 0$, that the value of $x_s = x_s^*$ determined by (11) is strictly positive.

Consider the feasible solution

$$\begin{aligned} x_i &= \bar{b}_i - \bar{a}_{is} x_s^* \geq 0 & (i = 1, 2, \dots, m) \\ (12) \quad x_s &= x_s^* = \bar{b}_r / \bar{a}_{rs} & (x_s^* > 0) \\ x_j &= 0 & (j = m+1, \dots, n) \\ & & (j \neq s). \end{aligned}$$

The total cost of such a solution is

$$(13) \quad z = z_0 + \bar{c}_s x_s^* < z_0.$$

There remains only to show that such a solution is a new basic feasible solution. It is clear from the definition of the index $i = r$ that

$$(14) \quad x_r = \bar{b}_r - \bar{a}_{rs} x_s^* = 0$$

and that we are trying to show that x_1, x_2, \dots, x_m (excluding x_r) and x_s constitute a new basic set of m variables. To prove this we simply observe that since $\bar{a}_{rs} > 0$, we may use the r -th equation of (4) and \bar{a}_{rs} as "pivot element" to eliminate the variable x_s from the other equations and the minimizing form. It is clear that we have again reduced our system to canonical form.

This ability to obtain a new canonical form from the previous one by means of a single additional elimination constitutes the key to the computational efficiency of the simplex method.

Iterative Procedure

The new basic feasible solution can now be tested for optimality. If not optimal then one may choose by criterion (9) a new variable x_s to increase and proceed to construct either: (a) a class of solutions in which $z \rightarrow -\infty$ as $x_s \rightarrow +\infty$ (if all $\bar{a}_{1s} \leq 0$) or (b) a new basic feasible solution in which the cost z is lower than the previous one (providing the values of the basic variables for the latter are strictly positive; otherwise the new value of z may be equal to the previous value of z).

The simplex algorithm consists of iterating the above "cycle" again and again, terminating only when there has been constructed:

- (a) a class of feasible solutions in which $z \rightarrow -\infty$; or
- (b) an optimal basic feasible solution.

If on each cycle the value of z decreases a non-zero amount, then the entire process will terminate in a finite number of cycles. The reason for this is there are only a finite number of ways to choose a set of m basic variables out of n variables. If the algorithm were to continue indefinitely, it could only do so by repeating the same basic set of variables—hence the same cost z .

Degeneracy

A basic solution is said to be "degenerate" if at least one of the values of the basic variables is zero. In this case it is clear by (11) that if for some $\bar{a}_{1s} > 0$, it happens that the corresponding value \bar{b}_1 of the basic variable is zero,

then no increase in x_g is possible (that will maintain the values of the basic variables non-negative). It follows also that there is no decrease in z for that iteration. In this case one can no longer argue that the procedure will terminate in a finite number of iterations because with no change in the value of z , it is conceivable that the same basic set of variables may reoccur, say, after k iterations. Hence if one were to continue with the same selection of s and r for each iteration as before, the same basic set would reoccur after $2k$ iterations and again after $3k$ iterations etc., indefinitely. If this happens it is referred to as "cycling" in the simplex algorithm. Theoretical examples of cycling have been constructed.

The high frequency of occurrence of degeneracy in practice is a curious phenomenon, for degeneracy can happen only when the values b_i of the original right-hand side in (1) bear a special relation to the coefficients of the basic variables. This is clear since the process of reduction to canonical form depends only on the coefficients and not on the right-hand side; the final values \bar{b}_i are weighted sums of the original b_i 's where the weights depend only on the coefficients. If the b_i were selected at random it would be something of a miracle if \bar{b}_i should vanish.

One way to place the entire theory on a rigorous foundation, is to give precise rules for slightly altering the values of the b_i so as to avoid degeneracy and thereby the possibility of cycling. However, discussion of this technique will be postponed to later. In practical applications the phenomenon of

cycling has never been observed in spite of the fact that degenerate basic solutions are frequently encountered. For this reason it is recommended that the algorithm be applied without any special rules to resolve the case of degeneracy.

Appendix: Systems of Linear Equations

Equivalent Systems: Since the simplex method for solving the linear programming problem depends on standard methods for solving a system of linear equations (which of course is a special case), we shall review properties of such systems.

Suppose that we have a system of m equations in n unknowns

(1). A solution of the i -th equation is a set of numbers $(x_1^i, x_2^i, \dots, x_n^i)$ such that

$$(15) \quad a_{i1}x_1^i + a_{i2}x_2^i + \dots + a_{in}x_n^i = b_i$$

A solution of the system (1) is a set of numbers which is a solution of every equation of the system.

Suppose there is another system of equations

$$(16) \quad \bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \dots + \bar{a}_{1n}x_n = \bar{b}_1$$

$$\bar{a}_{21}x_1 + \bar{a}_{22}x_2 + \dots + \bar{a}_{2n}x_n = \bar{b}_2$$

$$\bar{a}_{m1}x_1 + \bar{a}_{m2}x_2 + \dots + \bar{a}_{mn}x_n = \bar{b}_m$$

The two systems (1) and (16) are called equivalent if every solution of each is a solution of the other.

The process of solving a system of equations is that of

finding an equivalent system of simplest form.

Let us consider system (1). Let k_1, k_2, \dots, k_m be any numbers and let

$$(17) \quad \bar{a}_{l1}x_1 + \bar{a}_{l2}x_2 + \dots + \bar{a}_{ln}x_n = \bar{b}_l$$

be a linear equation formed by a linear combination of equations from (1), i.e., formed by multiplying the first equation by k_1 , the second by k_2 , etc. and summing the products. We are assuming

$$(18) \quad \begin{array}{l} \bar{a}_{l1} = k_1 a_{11} + k_2 a_{21} + \dots + k_m a_{m1} \\ \bar{a}_{ln} = k_1 a_{1n} + k_2 a_{2n} + \dots + k_m a_{mn} \\ \bar{b}_l = k_1 b_1 + k_2 b_2 + \dots + k_m b_m \end{array}$$

Consider a modified system of equations in which all equations are the same as (1) except the l -th equation is replaced by (17), i.e.,

$$(19) \quad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \bar{a}_{l1}x_1 + \bar{a}_{l2}x_2 + \dots + \bar{a}_{ln}x_n = \bar{b}_l \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}$$

Clearly every solution of (1) is a solution of system (19). Conversely let x'_1, x'_2, \dots, x'_n be any solution of (19). It is evidently the solution of every equation of (1) except possibly the l -th equation. But the substitution of (18) into the l -th equation of (19) reduces to

$$(20) \quad k_l (a_{l1}x'_1 + a_{l2}x'_2 + \dots + a_{lm}x'_m - b_l) = 0,$$

so if $k_l \neq 0$ it is also true that the l -th equation of (1) is satisfied.

Theorem:

If in a system of equations (1) the l -th equation is replaced by the sum of equations of the system after multiplication by the numbers k_1, k_2, \dots, k_m , where $k_l \neq 0$, then the new system is equivalent to the given system.

All methods of solving a system of equations employ the above principle.

Elementary Operations: There are two kinds of elementary operations which can be performed upon the equations of a system to yield an equivalent system.

Type I: Replace any equation

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

by

$$k(a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n) = kb_1$$

where $k \neq 0$.

Type II: Replace any equation

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

by

$$a_{11}x_1 + \dots + a_{1n}x_n + k(a_{l1}x_1 + \dots + a_{ln}x_n) = b_1 + kb_l$$

where $1 \neq l$.

Equivalent Triangular and Canonical $m \times m$ Systems: A system of m equations in m unknowns is triangular if it has the form

$$\begin{aligned}
 & a_{11}x_1 & = b_1 \\
 (21) \quad & a_{21}x_1 + a_{22}x_2 & = b_2 \\
 & a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & = b_3 \\
 & - - - - - \\
 & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mm}x_m = b_m,
 \end{aligned}$$

where the coefficients along the diagonal are non-vanishing,

$$(22) \quad a_{ii} \neq 0, \quad i = 1, 2, \dots, m.$$

It is also called triangular if by rearrangement of the order of the equations, and relabeling of the variables, it can be written in this form.

Not every system of m equations in m unknowns is equivalent to a triangular system. This is the case, as we shall see, if the original system contains a redundant equation. Also, if the original system has no solution, i.e., is inconsistent, then it could not be equivalent to a triangular system, since solutions for the latter can be readily obtained. For example, the first equation yields the value $x_1 = b_1/a_{11}$. When this value is substituted into the equations $i = 2, \dots, m$ the values of the remaining variables can be obtained by solving a triangular system in $m - 1$ equations in $m - 1$ unknowns. Accordingly, the problem can be solved by the process of successive substitutions. This process is really the same as a sequence of elementary operations, i.e., multiplying the first equation by $-a_{i1}/a_{11}$ and adding to the i -th equation replaces the latter by an equation in which the variable x_1 is eliminated, $i = 2, 3, \dots, m$. In a similar manner

x_2 may be eliminated from all but the second equation. This yields the simplest form of an equivalent system

$$(23) \quad \begin{aligned} x_1 &= \bar{b}_1 \\ x_2 &= \bar{b}_2 \\ &\vdots \\ x_m &= \bar{b}_m. \end{aligned}$$

This is a special case of the "canonical" form for linear programming. In order to solve a system of m equations in n unknowns it is common practice to try first to "reduce" it to an equivalent triangular system by means of elementary operations and then by successive substitutions to canonical form (23). In linear programming type work, however, it is more convenient to reduce it directly to canonical form (if possible).

Reduction of m by n system to an equivalent canonical system: A system of m equations in n ($m \leq n$) unknowns is in canonical form for linear programming if it can be written

$$(24) \quad \begin{aligned} x_1 &+ \bar{a}_{1,m+1}x_{m+1} + \dots + \bar{a}_{1j}x_j + \dots + \bar{a}_{1n}x_n = \bar{b}_1 \\ x_2 &+ \bar{a}_{2,m+1}x_{m+1} + \dots + \bar{a}_{2j}x_j + \dots + \bar{a}_{2n}x_n = \bar{b}_2 \\ &\vdots \\ x_m &+ \bar{a}_{m,m+1}x_{m+1} + \dots + \bar{a}_{mj}x_j + \dots + \bar{a}_{mn}x_n = \bar{b}_m \end{aligned}$$

The standard procedure for reducing (if possible) a general system (1) of m equations in n unknowns to equivalent canonical form will now be discussed.

Select any term $\bar{a}_{ij}x_j$ in system (1) such that $\bar{a}_{ij} \neq 0$. Call this the pivot element. Rearrange the order of the equations and

the variables so that this pivot element becomes $a_{11}x_1$.

The elementary operation of multiplying the first equation by $-a_{i1}/a_{11}$ and adding to the i -th equation, $i = 2, 3, \dots, m$, will eliminate x_1 from the remaining. Multiply the first equation by $1/a_{11}$.

Let a'_{ij} denote the coefficients in the new system. Consider next the system in $n-1$ unknowns x_2, x_3, \dots, x_n consisting of the last $m-1$ equations and repeat the process of selecting a pivot element, rearranging variables and equations so that the pivot element is $a'_{22}x_2$. Multiplying the second equation by $-a'_{i2}/a'_{22}$ and adding to the i -th equation for all i except $i = 2$ produces a system in which x_2 has been eliminated from all equations but the second, and x_1 from all but the first.

Continue in this manner until m variables are selected for the elimination process or until after selecting r variables it is not possible to find a pivot element among the remaining $m-r$ equations and $n-r$ variables. At this stage the original system has been reduced to the equivalent system

$$\begin{array}{rcl}
 x_1 & + \bar{a}_{1r+1}x_{r+1} + \bar{a}_{1r+2}x_{r+2} + \dots + \bar{a}_{1n}x_n & = \bar{b}_1 \\
 x_2 & + \bar{a}_{2r+1}x_{r+1} + \bar{a}_{2r+2}x_{r+2} + \dots + \bar{a}_{2n}x_n & = \bar{b}_2 \\
 & \text{---} & \\
 x_r & + \bar{a}_{rr+1}x_{r+1} + \dots + \bar{a}_{rn}x_n & = \bar{b}_r \\
 & 0 \ x_{r+1} + \dots + 0 \ x_n & = \bar{b}_{r+1} \\
 & 0 \ x_{r+1} & + 0 \ x_n = \bar{b}_m
 \end{array}$$

where the variables in (25) are some rearrangement of the order of the variables of the original system and the equations in (25) correspond to some rearrangement of the order of the original equations. The number r is called the rank of the system.

The original system has no solution, i.e., is inconsistent unless

$$(26) \quad \bar{b}_{r+1} = \bar{b}_{r+2} = \dots = \bar{b}_m = 0.$$

If relations (26) hold, this means that each of the $m-r$ equations of the original system corresponding to equations $r+1, \dots, m$ of the reduced system, is redundant, i.e., is equal to a linear combination of other equations of the system. In this case one can solve the system by choosing any values for $x_{r+1}, x_{r+2}, \dots, x_n$ and then determining values for x_1, x_2, \dots, x_r .

Only in the case where the original system is neither inconsistent nor redundant can the system be reduced to equivalent canonical form. Here the rank $r = m$. The set of m variables in the original system corresponding to x_1, x_2, \dots, x_m is called (as noted earlier) a basic set of variables.

II. THE SIMPLEX TABLEAU: AN EXAMPLE

The "simplex tableau" is a convenient way to present the canonical system in detached coefficient form in which the original order of the variables and the equations is kept (instead of rearranging them each time). A simple example will suffice to illustrate the idea.

Consider the problem of minimizing z where

$$\begin{aligned}
 &5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 = 20 \\
 (27) \quad &\textcircled{x_1} - x_2 + 5x_3 - x_4 + x_5 = 8 \quad (x_j \geq 0) \\
 &x_1 + 6x_2 - 7x_3 + x_4 + 5x_5 = z.
 \end{aligned}$$

We assume we know that x_1 and x_5 can be used as basic variables and the basic solution will be feasible. Accordingly we proceed to use the term x_1 as pivot element in say, the second equation (the circled element of (27)) to eliminate x_1 from the other equations: this yields

$$\begin{aligned}
 &x_2 - 12x_3 + 3x_4 - \textcircled{4x_5} = -20 \\
 (28) \quad &x_1 - x_2 + 5x_3 - x_4 + x_5 = 8 \\
 &x_2 - 12x_3 + 2x_4 + 4x_5 = z - 8.
 \end{aligned}$$

Using the term $-4x_5$ as pivot element in the first equation to eliminate x_5 , yields

$$\begin{aligned}
 &-\frac{1}{4}x_2 + 3x_3 - \frac{3}{4}x_4 + x_5 = 5 \\
 (29) \quad &x_1 - \frac{3}{4}x_2 + 2x_3 - \frac{1}{4}x_4 = 3 \\
 &8x_2 - 24x_3 + 5x_4 = z - 28,
 \end{aligned}$$

which, of course, is in canonical form except that we have not bothered to rearrange the rows and columns. The corresponding tableau is

(30)

1st Basic Solution		Variables				
Variable	Value	x_1	x_2	x_3	x_4	x_5
x_5	5		$-\frac{1}{4}$	3	$-\frac{3}{4}$	1
x_1	3	1	$-\frac{3}{4}$	(2)	$-\frac{1}{4}$	
-z	-28		8	-24	5	
		x	*			x

where the coefficients of x_1, x_2, x_3, x_4, x_5 are shown in detached coefficient form. The basic set of variables (x_1, x_5) is recorded in the first column in the order corresponding to the position of the 1 in the column of coefficients of x_1 and x_5 respectively. The symbol -z for the negative of the total cost is recorded at the bottom of this column. The constant terms (which are also the values of the basic solution and the negative total cost) are recorded in the "value" column. It is helpful when iterating the simplex algorithm to place check or x marks at the bottom of the columns corresponding to the variables in the basic set and a star in the column with the most negative relative cost factor (in this case $\bar{c}_3 = -24$).

It is also convenient to place a circle around that coefficient \bar{a}_{rs} of x_s which will be used as pivot element to obtain the next improved basic solution. In this case $\bar{a}_{23} = 2$ is the pivot element

because the smallest of the positive ratios formed by dividing the numbers \bar{b}_1 in the "value" column by the coefficients \bar{a}_{1s} that appear in the " x_s " column occurs for the ratio $x_3^* = \bar{b}_2/\bar{a}_{23} = 3/2$ (i.e., $3/2$ is the smallest of the ratios $5/3$, $3/2$). This means that x_3 will replace the basic variable that appears on the second row, x_1 , in the next cycle as a new basic variable. Indeed, eliminating x_3 from all but the second equation, the next tableau becomes

(31)

2nd Basic Solution		Variables				
Variables	Value	x_1	x_2	x_3	x_4	x_5
x_5	$1/2$	$-3/2$	$+7/8$		$-3/8$	1
x_3	$3/2$	$1/2$	$-3/8$	1	$-1/8$	
$-z$	8	12	-1		2	
			*	x		x

It will be noted that the relative cost factor for x_2 is negative. The basic solution is $x_3 = 3/2$, $x_5 = 1/2$ and $x_1 = x_2 = x_4 = 0$. The total cost is given by $-z = 8$ or $z = -8$. Since the solution is not optimal we again seek an improved basic solution. The smallest positive ratio occurs this time on row 1; hence, $7/8$ becomes the new pivot element. Thus in the next tableau x_2 replaces x_5 as a basic variable.

(32)

3rd Basic Solution		Variable				
Variable	Value	x_1	x_2	x_3	x_4	x_5
x_2	$4/7$	$-12/7$	1		$-3/7$	$8/7$
x_3	$12/7$	$-1/7$		1	$-2/7$	$3/7$
$-z$	$60/7$	$72/7$			$11/7$	$8/7$
			x	x		

Since all relative cost factors are nonnegative, the solution shown is minimal. The total cost is $z = -60/7$.

III. FINDING AN INITIAL BASIC FEASIBLE SOLUTION

Up to the present we have been assuming that a basic set of variables could be specified which could be used to perform the necessary initial eliminations and reduce the problem to canonical form. It has also been assumed that the associated basic solution is feasible.

It is true that many problems encountered in practice often have a starting solution readily at hand. For example, for the important class called "transportation" problems one can construct immediately a great variety of starting basic solutions. Economic models often contain storage and slack activities which permit an obvious starting solution in which nothing but these

1-2-50
1-9-50
-53-

activities take place. Such solutions may be a long way from the optimum solution but at least it is an easy start. Usually, little effort is required in these cases to reduce the problem to canonical form.

On the other hand many problems encountered in practice do not provide any obvious starting basic set of variables. In fact, little or nothing may be known (mathematically speaking) about the problem requiring solution:

(a) It may have redundancies -- for example, the equation balancing money flow may have been obtained from the equations balancing material flows by multiplying price by quantity and summing.

(b) It may have inconsistencies due to outright clerical errors or to use of inconsistent data.

(c) It may have impossible requirements considering the resources available. Thus it may be a problem in which resources are known to be in short supply and the main question asked is really whether a feasible solution exists at all.

From the above considerations it is clear that a general mathematical technique must be developed to solve linear programming problems free of any prior knowledge or assumptions about the systems being solved. In fact, if there are inconsistencies or redundancies these are important facts to be discovered.

A simple device will be presented in this section which uses the simplex algorithm itself to provide (if it exists) a

starting basic feasible solution. This part of the process is usually referred to as phase I. The second part of the process, obtaining an optimal basic feasible solution, is then referred to as phase II. The device has several important features that should be noted:

(a) No assumptions are made regarding the original system; it may be redundant, inconsistent, or not solvable in non-negative numbers.

(b) No eliminations are required to obtain an initial solution in canonical form for phase I.

(c) The end product of phase I is a basic feasible solution (if it exists) in canonical form ready to initiate phase II.

The procedure for phase I is as follows:

Step I: Arrange the original system of equations so that all constant terms b_i are positive (or zero) by changing, where necessary, the signs on both sides of any of the equations.

Step II: Augment the system to include a basic set of "artificial" variables $x_{n+1} \geq 0$, $x_{n+2} \geq 0$, ..., $x_{n+m} \geq 0$ so that

$$\begin{aligned} (33) \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} = b_1 \quad (b_1 \geq 0) \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + x_{n+2} = b_2 \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + x_{n+m} = b_m \end{aligned}$$

and

$$(34) \quad x_j \geq 0 \quad (j = 1, 2, \dots, n; n+1, \dots, n+m).$$

Step III: Using the simplex algorithm, find a solution to (33) and (34) which minimizes the sum of the artificial variables,

denoted by w . We call this the "infeasibility form"

$$(35) \quad x_{n+1} + x_{n+2} + \dots + x_{n+m} = w.$$

Step IV: If $\text{Min } w > 0$, then no feasible solution exists and the procedure is terminated. On the other hand, if $\text{Min } w = 0$, then all $x_{n+1} = 0$ and the basic feasible solution constitutes a feasible solution to the original unaugmented system. Initiate phase II of the simplex algorithm by

(a) dropping from further consideration all non-basic variables whose relative cost factors for form w are strictly positive.

(b) replacing the linear form w (as modified by various eliminations) by the linear form z , after first eliminating from z all basic variables. It is common computational practice to perform the elimination of the basic variables from z on each cycle of phase I. In this case, the modified form z may be used immediately to initiate phase II.

Discussion: While the procedure for phase I is almost self-evident, it does deserve some discussion. It is clear that if there exists a feasible solution to the original system (1) and (2), then this same solution also satisfies (33) and (34) with the artificial variables set equal to zero. Thus, $w = 0$ in this case. From (35), the smallest possible value for w (since it is the sum of non-negative variables) is zero. Hence, if feasible solutions exist, the minimum value of w will be $w = 0$; conversely, if a solution is obtained for (33) and (34) with $w = 0$, it is clear that all $x_{n+1} = 0$ and the values of x_j

for $j \leq n$ constitute a feasible solution to (1). It also follows that if $\text{Min } w > 0$, then no feasible solutions to (1) exist.

It should be noted that whenever the original system contains redundancies (and often when degenerate solutions occur), artificial variables will appear as part of the basic set of variables in phase II. If they do appear it is necessary that their values never exceed zero. This is accomplished in Step IVa where all non-basic variables are dropped whose relative cost factors for w (denoted by \bar{d}_j) are positive. The justification for this procedure is based on theorems 1 and 2. For clarity we shall repeat the argument. The basic variables at completion of phase I will appear in the reduced form w with zero coefficients; in fact, the form of w after elimination of the basic variables will be

$$(36) \quad \bar{d}_1 x_1 + \bar{d}_2 x_2 \dots \bar{d}_{n+m} x_{n+m} = w - w_0 \quad (\bar{d}_j \geq 0; w_0 = 0)$$

where $\bar{d}_j = 0$ for basic variables and all $\bar{d}_j \geq 0$ because the solution is optimal (theorem 1); also $w_0 = \text{Min } w = 0$ since feasible solutions exist. All feasible solutions to (1) are feasible solutions to (36) with $x_{n+1} = 0$ and $w = 0$. They cannot contain an $x_j > 0$ with $\bar{d}_j > 0$ because this implies $w > 0$; hence, all such x_j may be dropped from further consideration. If we drop them, then we confine our attention, as in phase II, only to variables whose corresponding $\bar{d}_j = 0$. By (36) solutions involving only these variables have $w = 0$, and consequently are feasible for the original problem. Thus

Theorem 3:

If artificial variables form part of the basic sets of variables in the various cycles of phase II, their values will never exceed zero.

To illustrate, let us return again to example (27). Since the constant terms are all positive we augment the system with the auxiliary variables x_6 and x_7 and consider for phase I

$$\begin{aligned} 5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 + x_6 &= 20 \quad (x_j \geq 0) \\ (37) \quad x_1 - x_2 + 5x_3 - x_4 + x_5 + x_7 &= 8 \\ x_6 + x_7 &= w. \quad (w = \text{Min}) \end{aligned}$$

Eliminating the basic variables x_6 and x_7 from w by adding the first two equations and subtracting from w yields the starting tableau I-0 (below) for phase I. Applying iteratively the simplex algorithm yields tableaux I-1, I-2, and II-0 which is the start of phase II.

I-0	Basic Solution		Variables						
	Symbol	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7
	x_6	20	5	-4	13	-2	+1	1	
	x_7	8	1	-1	+5	-1	+1		1
	$-w$	-28	-6	5	-18	+3	-2		
					*			X	X

I-1	x_3	1.54	.39	-.31	1	-.15	.08	.08	
	x_7	.30	-.92	+.54		-.23	+.62	-.39	1
	-w	-.31	+.92	-.54		+.23	-.61	1.39	1
					X		*		X

I-2	x_3	1.5	.5	-.375	1	-.125		+.125	-.125
	x_5	.5	-1.5	+.875		-.375	1	-.625	1.625
	w	0						1	1
					X		X		

II-0	x_1	1.5	.5	-.375	1	-.125	
	x_3	.5	-1.5	+.875		-.375	1
	-z	8	12	1		2	
					X		X

Optimal)

On the first iteration the value of w was reduced from 28 to .31, on the next iteration $w = 0$ and a feasible basic solution $x_1 = 1.5$, $x_3 = .5$ is obtained for the original unaugmented system. Variables x_6 and x_7 have positive relative cost factors for w and hence must be dropped for phase II. Tableau II-0 is obtained from I-2 by dropping columns associated with these variables, and replacing the w row by a z row using the first two rows of I-2 to eliminate x_3 and x_5 from z . Because of the canonical form of

I-2 this elimination is accomplished by multiplying the first and second rows by the coefficients of x_3 and x_5 respectively in form z and subtracting from z . By good luck all relative cost factors of form z are positive in II-0; the starting feasible solution for phase II turns out also to be the optimal solution (a happy accident) and therefore no further iterations are required.